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FAST TRACK COMMUNICATION

A family of many-body models which are exactly solvable analytically

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Abstract
We present a family of many-body models which have an exact analytical solution. Surprisingly, these models include generalizations of such interesting physical systems as Bose–Einstein condensates with Josephson-type interactions. The generalization comprises the inclusion of inelastic collisions, which are present in real systems but are not accounted for in the canonical model. The unexpected insight of our paper is that the inclusion of these additional terms can render the system exactly solvable. Our results open up an arena to study many-body system properties analytically, where hitherto numerical studies had to be employed.

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Many-body systems are of great relevance in most areas of physics. In particular, there has been increasing interest in studying the properties of many-body systems and learning how to manipulate them in order to implement quantum information processing [1]. Ion traps, NMR systems, optical lattices, spin chains and many others have been investigated for this purpose [2]. Unfortunately, interesting many-body systems are rarely accessible to purely analytic analysis. Exact solutions exist mainly for one-dimensional systems, but the higher dimensional cases have to be treated numerically [3]. Numerical calculations are in practice limited by the growing degrees of freedom of the system. In nuclear physics, the Lipkin–Meshkov–Glick (LMG) model [4] was introduced as a toy model to study many-body properties. It has been studied extensively because its integrability allows for numerical analysis [5] and approximate solutions using the algebraic Bethe ansatz [6]. Recently, it has been used to find interesting results on entanglement in many-body systems [7]. The LMG model has relevance in quantum optics since it is related to the two-mode Bose–Einstein condensate (BEC) [8]. In the study of BECs, multicomponent condensates are of main interest. However, the lack of analytical

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solutions has restricted our understanding of such mesoscopic systems, whose most intriguing property is their collective quantum behaviour.

In this paper, we present a family of many-body models which are solvable analytically. The members of this family are labelled by the integer parameter n which indicates the maximum number of particles which interact in the system; the n-model considers the n-body, $(n-1)$-body, ..., and 2-body interactions. A possible realization of the model considers N spin-1/2 interacting particles in the presence of a classical coherent field which coherently manipulates the state of the system in order to, for example, process quantum information. In particular, the $n = 2$ model corresponds to an extended LMG model which considers single spin-flip terms produced by the interaction with the field and the effects of particle interactions during the spin-flip process. This extension allows for an exact analytical solution of the model.

Another realization of the models is a two-mode Bose–Einstein condensate with a Josephson-type interaction. The canonical Josephson Hamiltonian [8] considers elastic two-particle collisions and has no analytical solution. So far, the canonical two-mode BEC model has not been quantitatively verified by experiment although it qualitatively describes some of the observed effects [9]. Our exactly soluble n-model provides a more complete description of a two-mode BEC since it considers all the features of the previously mentioned model, but in addition considers all n-particle elastic and inelastic collisions. Indeed, these multi-particle collisions must be included in a more realistic description of the condensate: it has been extensively pointed out by experiment that inelastic collisions [10] are present in the BEC and play an important role in some systems. Moreover, higher order collisions are relevant beyond the dilute regime of BECs [9, 11]. So far only a couple of models incorporate non-elastic collisions in multi-mode condensates [12]. Many-particle collisions have not been addressed at all in the theoretical models, although it is known that they are physically relevant especially in the coldest phase of the condensate where the particle density is high [11]. Ironically, an effort is purposely made in the laboratory to suppress many-particle collisions and inelastic processes, in order to allow for comparison with the existing theoretical models [9]. Here, we show that including these processes in the theoretical description the model becomes exactly soluble.

We show, using the 2-model, that the evolution of the relative population of the condensate presents collapse and revivals of Rabi oscillations. We calculate the ground state of the system, and show that under certain circumstances the ground state is in a multiple macroscopic superposition of coherent states. The analytical solutions of this family of models will allow for a deeper understanding of many-body properties.

We introduce our models by considering the family of Hamiltonians $H_0^n = \sum_{i=0}^n A_i J_z^i$ where J_z is some representation of the $SU(2)$ angular momentum operator in the z-direction and A_i are real constants. Since $J_z|j,m\rangle = m|j,m\rangle$ with j and m integers or half integers with $m = -j, -j+1, \ldots, j-1, j$, the eigenstates of the Hamiltonian are $|j,m\rangle$ with energy $\mathcal{E}_m^n = \sum_{i=0}^n A_i m^i$. By applying $U = \mathrm{e}^{\mathrm{i}\phi J_z}\,\mathrm{e}^{\mathrm{i}\theta J_y}$, which is the most general rotation of J_z in the $SU(2)$ algebra with angles ϕ and θ, to H_0^n we construct the family of n-models,

$$H^n = U^\dagger H_0^n U = \sum_{i=0}^{n} A_i (U^\dagger J_z U)^i. \tag{1}$$

The exact and analytical solution of these Hamiltonians is of course simply $U^\dagger|j,m\rangle$ with energy $\mathcal{E}_m^n$. The integer parameter n defines the n-model by considering up to n powers of J_z in the H_0^n Hamiltonian. In terms of $J_\pm = J_x \pm \mathrm{i}J_y$, the 2-model is

$$\begin{aligned} H_2 = {} & A_1(\cos\theta J_z + \sin\theta(\mathrm{e}^{\mathrm{i}\phi}J_+ + \mathrm{e}^{-\mathrm{i}\phi}J_-)/2) + A_2\big(\cos^2\theta J_z^2 + \sin^2\theta\big(\mathrm{e}^{2\mathrm{i}\phi}J_+^2 + \mathrm{e}^{-2\mathrm{i}\phi}J_-^2 \\ & + J_+J_- + J_-J_+\big)/4 + \cos\theta\sin\theta(J_z(\mathrm{e}^{\mathrm{i}\phi}J_+ + \mathrm{e}^{-\mathrm{i}\phi}J_-)/2 + \mathrm{h.c})\big). \end{aligned} \tag{2}$$

This Hamiltonian may be written in terms of two bosonic operators $[a, a^\dagger] = [b, b^\dagger] = 1$ through the Schwinger representation which relates the bosonic operators to the angular momentum ones in the following way: $J_z = (a^\dagger a - b^\dagger b)/2$, $J_+ = a^\dagger b$ and $J_- = ab^\dagger$. Thus, the state $|j, m\rangle$ is identified with $|n_a = j + m\rangle \otimes |n_b = j - m\rangle$. Note that the commutation relations of the $SU(2)$ operators are indeed satisfied, and the total number operator $N = n_a + n_b = a^\dagger a + b^\dagger b$ is related to the total angular momentum by $J = N$. Therefore, by choosing different representations of the $SU(2)$ operators one can vary the total number of particles N. The 2-model in the two-mode representation is

$$H_2 = A_0 + \delta\omega(a^\dagger a - b^\dagger b) + \lambda(\mathrm{e}^{\mathrm{i}\phi} a^\dagger b + \mathrm{e}^{-\mathrm{i}\phi} ab^\dagger) + \mathcal{U} a^\dagger b^\dagger ab + \Lambda(\mathrm{e}^{2\mathrm{i}\phi} a^\dagger a^\dagger bb + \mathrm{h.c.}) + \mu((a^\dagger a^\dagger ab - b^\dagger a^\dagger bb)\,\mathrm{e}^{\mathrm{i}\phi} + \mathrm{h.c.}), \quad (3)$$

with $A_0 = A_2(\cos^2\theta N^2 + \sin^2\theta N)/4$, $\delta\omega = (A_1\cos\theta)/2$, $\mathcal{U} = A_2(1 - 3\cos^2\theta)/2$, $\lambda = (A_1\sin\theta)/2$, $\mu = (A_2\cos\theta\sin\theta)/2$ and $\Lambda = (A_2\sin^2\theta)/4$. Note that the particles are indistinguishable and the model only accounts for how many of them are in a given state. We will devote the rest of this paper to show that the family of Hamiltonians given by equation (1) describes real physical situations of great interest. Despite the strikingly simple mathematical form, the physical content of the models is rich.

The n-model describes the n-body, $(n - 1)$-body, ..., and 2-body interactions of $N = a^\dagger a + b^\dagger b$ spin-1/2 particles (with $n \leqslant N$) in the presence of a classical coherent field. We will first analyse the 2-model. The first term in equation (3) describes the free energy of $a^\dagger a$ spin-1/2 particles in the spin-up state and $b^\dagger b$ in the spin-down state with frequency difference $\delta\omega$. The interaction between two spins has strength $\mathcal{U}$ and corresponds to a dispersive process in which spins exchange their state while total spin is conserved. Additionally, we consider the interaction with an external classical field that produces one spin to flip state with coupling constant λ. The classical field could be an effective field due to the presence of another system, other degrees of freedom of the system or, more interesting, to an external experimenter manipulating the state of the system using a laser. This last situation would be necessary for manipulation quantum information in the system.

Due to the interaction of the field with the system, there is also a probability, parameterized by μ, of having two spins flip their state. Since the Hamiltonian has a second-order character, i.e. it considers products of two and four creation and annihilation operators, one must consistently consider all possible second-order physical processes. Thus, we include the two-particle spin-flip term (Λ) and the term that describes a single dispersive process (μ) taking place due to particle interaction while the laser produces a single spin to flip. It is remarkable that considering these extra terms allows for an analytical solution of the system.

Now we can analyse what the n-model describes all possible n-body, $(n - 1)$-body, ..., and 2-body interactions with $n \leqslant N$. Thus, equation (1) describes the possibility of n spin-1/2 particles exchanging their state in such a way that the total spin is conserved and considers a laser which causes $m \leqslant n$ particles to flip state and all the possible dispersion terms accompanying this process.

If no classical field is applied, the model simply consists of N spin-1/2 particles which interact in such a way that the total spin is always conserved. The spin is conserved because no energy is provided to the system.

Surprisingly, ignoring the single spin-flip term (λ) and the term with a spin-flip plus single dispersion (μ), the 2-model corresponds to the LMG model of nuclear physics. The LMG model was constructed using products of two and four creation and annihilation operators with the purpose of creating a simple model for testing many-body properties. It has so far no physical realization and it does not admit an exact analytical solution. Here, we showed that considering an extension to the model, by considering consistently all possible products of two and four creation and annihilation operators, an exact solution is found.

Now let us focus on a closely related problem which does have a physical realization. We can interpret the family of Hamiltonians in equation (1) as a two-mode BEC with a Josephson-type interaction. The modes $a^\dagger, a$ and $b^\dagger, b$ with frequency difference $\delta\omega$ correspond to either atoms with two different hyperfine levels [13] or, alternatively, two spatially separated condensates [14]. The Josephson-type interactions is induced by applying a laser [13] or a magnetic field gradient [14]. In our Hamiltonian, the Josephson-type term, in which one particle is annihilated in one mode and created in the other, has coupling constant λ and phase ϕ. The terms with four bosonic operators describe two-particle elastic and inelastic collisions. The elastic collisions have interaction strength $\mathcal{U}$. The inelastic collisions have interaction strength μ when two particles in the same mode collide and one of them is transformed into the other and interaction strength Λ when the collision transforms two particles in one mode into the other.

Note that by fixing $\delta\omega$, λ and U the inelastic collision constants, Λ and μ, are determined. This is because in our model the inelastic collisions are produced by the effect of the Josephson-type interaction on colliding particles. It has been observed in experiments that the rate of inelastic collisions between atoms is increased when there is an interaction with a laser field [15]. This physical relationship is mathematically expressed by the relationship between the coefficients. In our model, the rate of elastic to inelastic collisions is given by

$$\frac{\mu + \Lambda}{\mathcal{U}} = \frac{\lambda}{2}\left(\frac{\lambda + 2\delta\omega}{\lambda^2 - 2\delta\omega^2}\right). \tag{4}$$

Ignoring the inelastic terms in equation (3), we find that our model coincides with the canonical Josephson Hamiltonian [8] when the rate of collisions of same particle type is equal. The assumption of equal collision rates for the same particle type is also made in [8] order to find approximate and numerical solutions. Our model has the same number of free parameters as the canonical two-mode Hamiltonian. The only difference is that Hamiltonian in equation (3) includes inelastic collisions, which are usually present in real BECs [10]. In magnetic traps, inelastic collisions are commonly suppressed to a large extent because they give rise to atom losses. Particle-type exchange is the dominant loss mechanism in two-mode condensates [16]. But in optical traps the particle loss due to particle-type exchange is negligible and there it is no longer necessary to suppress the process [16]. Including the correct rate of inelastic collisions in the Hamiltonian allows for an analytical solution which is simply $U^\dagger|N, m\rangle$. Note that U in the Schwinger representation is the two-mode displacement operator $U = \mathrm{e}^{\xi a^\dagger b - \xi^* a b^\dagger}$ where $\xi = \frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}$. In [19], an author of this paper and collaborators proposed H_2 to generate Berry phases in BECs without understanding that this could indeed correspond to a solution of the two-mode BEC problem.

We would like to emphasize that if for a specific physical system the rates of inelastic to elastic collisions do not hold or cannot be arranged by external manipulation, an analytical solution cannot be found using our method for such a condensate. Fortunately, in the laboratory the rate of elastic and inelastic collisions can be manipulated, for example, by applying a magnetic potential [17] making possible to meet the experimental values for the production ratios of those terms.

Let us now focus our attention on the 3-model Hamiltonian H_3. This Hamiltonian corresponds to all possible three-particle and two-particle interactions including elastic and inelastic collisions between the same and different particle type. The n-model in equation (1) describes a two-mode BEC where n-body, $(n-1)$-body, . . . , and 2-body interactions are

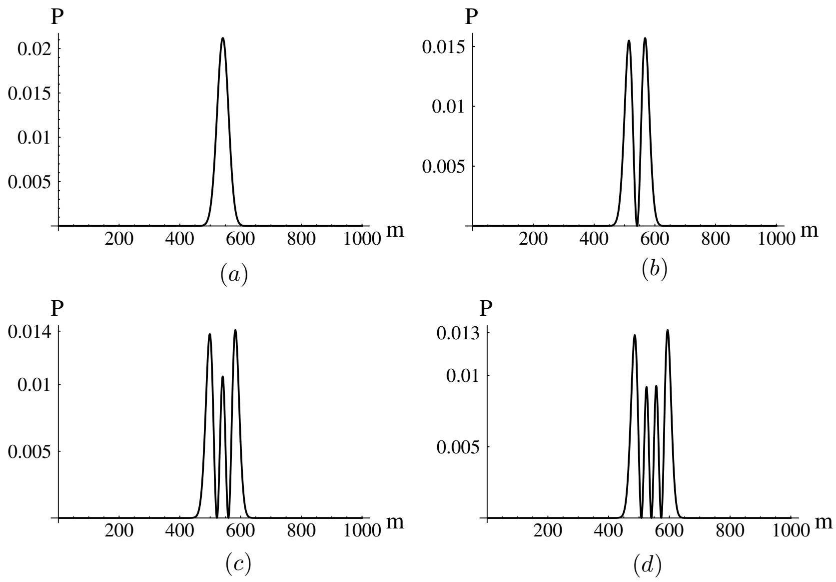

Figure 1. Ground state relative population distribution for 1000 atoms. Different m_0 correspond to different intensities of the laser. Quantum superposition appears when $m_0 < 1000$. (*a*) $m_0 = 1000$, (*b*) $m_0 = 999$, (*c*) $m_0 = 998$ and (*d*) $m_0 = 997$.

considered. In the two-mode BEC, *n*-particle collision terms are in principle present specially when the particle density is high.

The canonical two-mode model [8] considers only two-particle elastic collisions and has no exact analytical solution. Commonly, the Bethe ansatz is used to find the ground and first excited state solution, or numerical work is needed. The model introduced here is more general and has an exact analytical solution. It is possible to analyse the whole spectrum and one needs not to restrict the attention only to the ground state.

Due to the simplicity of our solution the ground state $U^\dagger|N, m_0\rangle$ of H_2 is trivially found by minimizing the energy $E_m^2 = A_1 m + A_2 m^2$ with respect to m. For $A_2 > 0, m_0$ is the nearest integer to $-A_1/(2A_2)$ or $m_0 = -A_1 N/|A_1|$ when $|-A_1/(2A_2)| > N$. For $A_2 < 0$, the minimum corresponds to $m_0 = N$ if $A_1 < 0$ or $m_0 = -N$ otherwise.

The solution for $A_2 < 0$ and $A_1 > 0$ is the coherent state which is the ground state solution of the LMG model in the limit of a large number of particles [18]. The canonical two-mode BEC predicts that the ground state of the condensate is, under certain conditions, a macroscopic superposition of two peaked distributions [8]. In figure 1, we plot the relative population distribution $P(m) = |\langle N, m|U^\dagger|N, m_0\rangle|^2$ for different ground states m_0 and find that macroscopic superpositions can involve several components for $m_0 \leqslant 1000$ as shown in figures 1(*c*) and (*d*). This difference must be due to inelastic collisions.

The evolution of the relative population J_z for a given initial state $|\psi(t = 0)\rangle = \sum_{m=-N}^{N} C_m U^\dagger|N, m\rangle$, with coefficients C_m, is given by

$$\begin{aligned}\langle J_z\rangle &= \cos\theta \sum_{-N}^{N} m|C_m|^2 - \sin\theta \sum_{-N+1}^{N} C_m C_{m-1} L_m \\ L_m &= \cos(\phi + (E_{m-1} - E_m)t)(N(N+1) - m(m-1))^{1/2}.\end{aligned} \tag{5}$$

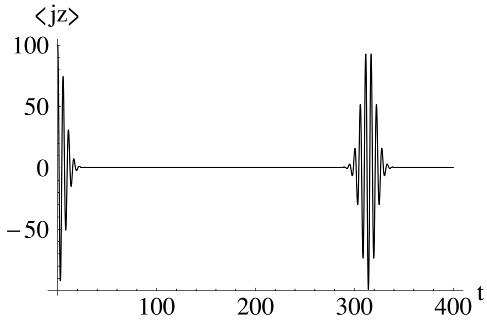

Figure 2. Evolution of $\langle j_z \rangle$. The initial condition is $|\psi(0)\rangle = |N = 100, m = 100\rangle$. $A_1 = 1$, $A_2 = 0.01$, $\theta = 3/2$.

The expectation value of J_y which describes the evolution of the relative phase of the condensates is equal to

$$\langle J_y \rangle = \sum_{-N+1}^{N} C_m C_{m-1} K_m \qquad K_m = \sin(\phi + (E_{m-1} - E_m)t)(N(N+1) - m(m-1))^{1/2}. \tag{6}$$

In figure 2, we plot the evolution of the relative population for the initial state $|N, N\rangle$ where the condensate consists of a single specie. The system presents Rabi-type oscillations with collapse and revivals. We are currently studying the effects of higher order collisions in the oscillations. Interesting generalizations of our family of models which are currently under study include a family of Hamiltonians with squeezing terms and models replacing the $SU(2)$ for $SU(3)$ algebra. The first generalization is performed by applying a two-mode squeezing operator $S(\alpha) = \mathrm{e}^{\alpha a^\dagger b^\dagger - \alpha^* ab}$ to the Hamiltonian H_0^n which then has solution $S^\dagger(\alpha)U^\dagger(\theta, \phi)|j, m\rangle$. The $SU(3)$ model is found by applying the most general rotation in the $SU(3)$ algebra to a polynomial in the $SU(3)$ diagonal generators. The bosonic representation of the $SU(3)$ model describes a three-mode (or spin-1) BEC. In principle, the model can be extended to the $SU(n)$ algebra corresponding to spin-J condensates. A general method of finding models with exact analytical solution can be extrapolated from our model. For a given algebra, find the diagonal generators, construct a polynomial in them and apply the most general rotation of the operators in the algebra to generate a new Hamiltonian.

The understanding of many-body systems in dimensions higher than 1 has been limited by the lack of any realistic models with analytical solutions. The model we have introduced here allows for the first time an analytical study of the n-body interactions of N spin-1/2 particles in the presence of a classical coherent field and a two-mode BEC with n-body elastic and inelastic collisions. The model extends the Lipkin–Meshkov–Glick model of nuclear physics and the canonical two-mode BEC models. The clear advantages of our model over these models include the possibility of studying higher order interactions. Currently, we study inelastic and many-body interactions in BECs and their effects in the phase transitions and entanglement properties of the system. We consider that the family of models that we have introduced opens an arena to study higher dimensional many-body systems analytically.

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